Minimum time trajectory with velocity constraints

Yunlong

- Problem statement

Consider the minimum time optimal control problem where we want to find an optimal command (or trajectory) that has minimum time T for a quadrotor to fly from an initial position x_0 with an initial velocity v_0 to a goal position x_T with a final velocity v_T. Using Pontryagin method to solve this constrained optimal control problem results in the well-known bang-bang control solution, in which the optimal solution for the control command is to always execute its extreme commands.

In addition to putting constraints on the input command, here we also add a constraint to the velocity such that the velocity of the quadrotor is limited.

$$
\begin{aligned}
&\min_u \int_0^T 1dt && \text{Find the optimal control variable that minimizes total time} \\
&s.t.\ \dot{x_1}(t) = x_2(t), \quad x_1(0) = a, x_1(T) = b && \text{Dynamics constraint} \\
&\quad \dot{x_2}(t) = u(t), \quad x_2(0) = c, x_2(T) = d && \text{Dynamics constraint} \\
&\quad -u_{min} \leq u \leq u_{max} && \text{Input constraint} \\
&\quad -v_{min} \leq v \leq v_{max} && \text{Velocity constraint}
\end{aligned}
$$

- Solution:

The problem above has a closed-form solution by using the Pontryagin minimum principle. It can be verified that the optimal control command $u^*(t)$ consists of ***at most 3 sections:***

- $[0, t_1]$ ***is an interior arc, with*** $u^* = u_{max}$ or $u^* = -u_{min}$.
- $[t_1, t_2]$ ***is a boundary arc, with*** $u^* = 0$.
- $[t_2, T]$ ***is an interior arc, with*** $u^* = -u_{min}$ or $u^* = u_{max}$.

Hence

- $u^*(t) = \begin{cases} +u_{max} & 0 \leq t < t_1 \\ 0 & t_1 \leq t < t_2 \\ -u_{min} & t_2 \leq t \leq T \end{cases}$

Case 1 & 2:

$$t_1 = \frac{-v_0(u_{max}+u_{min}) \pm \sqrt{(u_{max}+u_{min})(v_0^2 u_{min} + v_T^2 u_{max} - 2u_{max}u_{min}x_0 + 2u_{max}u_{min}x_T)}}{u_{max}(u_{max}+u_{min})}$$

$$t_2 = t_1$$

$$T = \frac{(u_{max}+u_{min})t_1 + (v_0 - v_T)}{u_{min}}$$

$$0 \le t_1 = t_2 \le T$$

Case 3:

$$t_1 = \frac{v_{max} - v_0}{u_{max}}$$

$$t_2 = \frac{u_{min}v_0^2 - 2u_{min}v_0 v_{max} - u_{max}v_{max}^2 + u_{min}v_{max}^2 + u_{max}v_T^2 - 2u_{max}u_{min}x_0 + 2u_{max}u_{min}x_T}{2u_{max}u_{min}v_{max}}$$

$$T = \frac{v_0 - v_T + u_{max}t_1}{u_{min}} + t_2$$

$$0 \le t_1 < t_2 \le T$$

- $u^*(t) = \begin{cases} -u_{min} & 0 \le t < t_1 \\ 0 & t_1 \le t < t_2 \\ +u_{max} & t_2 \le t \le T \end{cases}$

Case 4 & 5:

$$t_1 = \frac{v_0(u_{max}+u_{min}) \pm \sqrt{(u_{max}+u_{min})(v_0^2 u_{max} + v_T^2 u_{min} + 2u_{min}u_{max}x_0 - 2u_{min}u_{max}x_T)}}{u_{min}(u_{max}+u_{min})}$$

$$t_2 = t_1$$

$$T = \frac{(u_{min}+u_{max})t_1 + (v_T - v_0)}{u_{max}}$$

$$0 \le t_1 = t_2 \le T$$

Case 6:

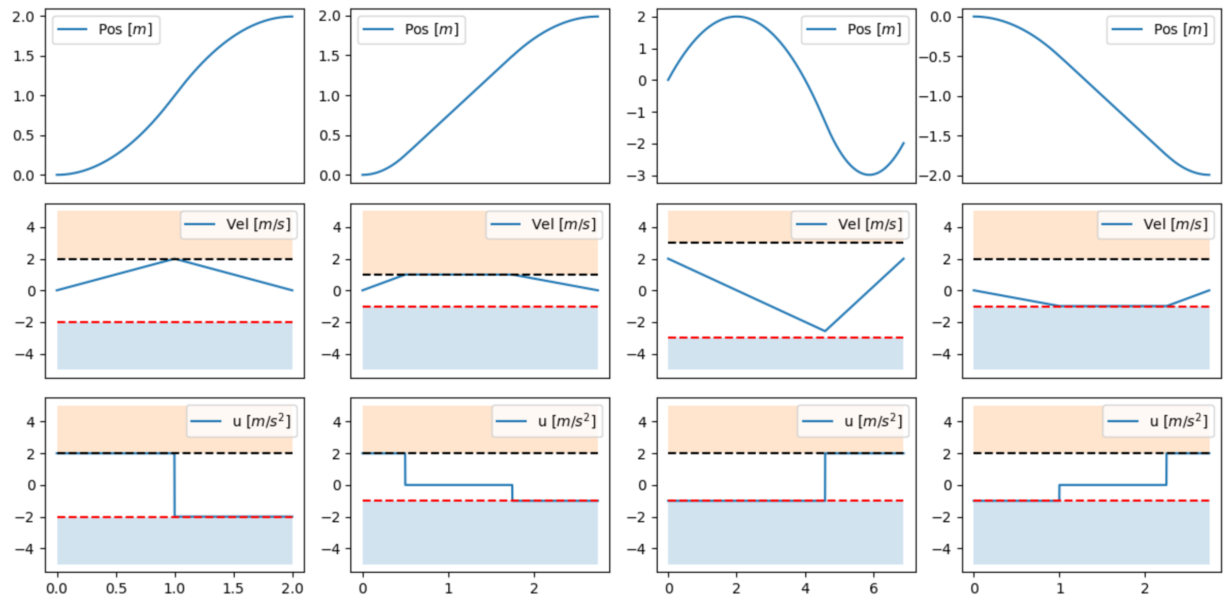
$$t_1 = \frac{v_0 + v_{min}}{u_{min}}$$

$$t_2 = \frac{u_{max}v_0^2 + 2u_{max}v_0 v_{min} + u_{max}v_{min}^2 - u_{min}v_{min}^2 + u_{min}v_T^2 + 2u_{max}u_{min}x_0 - 2u_{max}u_{min}x_T}{2u_{max}u_{min}v_{min}}$$

$$T = \frac{v_T - v_0 + u_{min}t_1}{u_{max}} + t_2$$

$$0 \le t_1 < t_2 \le T$$

- Examples



- Derivation:

Hamiltonian Function:

$$\begin{aligned} &\min_u \int_0^T 1 + \eta_1(v_{max} - x_2) + \eta_2(x_2 - v_{min})dt \\ H(x, u, p) &= g(x, u) + p^T f(x, u) \\ &= 1 + \eta_1(v_{max} - x_2) + \eta_2(x_2 - v_{min}) + p_1 x_2 + p_2 u \\ \text{where :}& \\ &\eta_1 \geq 0,\ \eta_1 = 0 \text{ if } v_{max} - x_2 > 0 \\ &\eta_2 \geq 0,\ \eta_2 = 0 \text{ if } x_2 - v_{min} > 0 \end{aligned}$$

Costate Equation:

$$\dot{p}_1(t) = -\frac{\partial H(x, u, p)}{\partial x_1} = 0 \quad (1)$$

$$\dot{p}_2(t) = -\frac{\partial H(x, u, p)}{\partial x_2} = -p_1 + \eta_1 - \eta_2 \quad (2)$$

The optimal control u^* is the control input that minimizes the Hamiltonian function:

$$\begin{aligned} u^* &= \text{argmin}_u H(x, u, p, \eta) \\ &= \text{argmin}_u \{p_2 u\} \end{aligned}$$

Maurer proves in Corallary that, for problems of this form, the adjoint variables p are continuous. Furthermore, $p_2 = 0$ must hold when a state constraint is ***boundary arcs.***

- **On interior arcs,** $\eta_1 = \eta_2 = 0$, $p_2 \neq 0 \text{ and } u^* = u_{max} \text{ or } u^* = u_{min}.$
- **On boundary arcs,** $x_2 = v_{max} \text{ or } x_2 = -v_{max}$ and it follows that $p_2 = 0 \text{ and } p_2 = 0$ and $u^* = 0.$
- According to Equation (2), the trajectory p_2 must be composed of **parabolic shapes** .

It can be verified from the above constraints that the trajectory consists of ***at most 3 sections:***

- $[0, t_1]$ ***is an interior arc, with*** $u^* = u_{max} \text{ or } u^* = -u_{min}.$
- $[t_1, t_2]$ ***is a boundary arc, with*** $u^* = 0.$
- $[t_2, T]$ ***is an interior arc, with*** $u^* = -u_{min} \text{ or } u^* = u_{max}.$

Now we can solve for optimal $\{t_1, t_2, T\}$ with the system dynamics and its boundary conditions

$$\begin{gathered} \dot{x}_1(t) = x_2(t) \\ \dot{x}_1(t) = u(t) \\ x_1(0) = a \qquad x_1(T) = b \\ x_2(0) = c \qquad x_1(T) = d \\ u(t) = u^*(t) \end{gathered}$$

$$u^*(t) = \begin{cases} +u_{max} & 0 \leq t < t_1 \\ 0 & t_1 \leq t < t_2 \\ -u_{min} & t_2 \leq t \leq T \end{cases}$$

$$
\begin{aligned}
t \sim [0, t_1) &: \\
&v_t = v_0 + u_{max} t \\
&x_t = x_0 + v_0 t + \frac{1}{2} u_{max} t^2 \\
t \sim [t_1, t_2) &: \\
&v_t = v_{t_1} + 0 * (t_2 - t_1) \\
&x_t = x_{t_1} + v_{t_1} (t_2 - t_1) + \frac{1}{2} * 0 * (t_2 - t_1)^2 \\
t \sim [t_2, T] &: \\
&v_t = v_{t_2} - u_{min} * (T - t_2) \\
&x_t = x_{t_2} + v_{t_2} (T - t_2) - \frac{1}{2} u_{min} (T - t_2)^2
\end{aligned}
$$

Now, lets solve the equations. As you can see, here we have 3 unknowns t_1, t_2, T .

1) :

$$
\begin{aligned}
v_{t_1} &= v_0 + u_{max} t_1 \\
x_{t_1} &= x_0 + v_0 t_1 + \frac{1}{2} u_{max} t_1^2
\end{aligned}
$$

2) :

$$
\begin{aligned}
v_{t_2} &= v_0 + u_{max} t_1 \\
x_{t_2} &= x_0 + v_0 t_1 + \frac{1}{2} u_{max} t_1^2 \\
&\quad + (v_0 + u_{max} t_1)(t_2 - t_1)
\end{aligned}
$$

3) : Finally, we get following two equations with 3 unknowns, which is not solvable.

$$
\begin{aligned}
v_T &= v_0 + u_{max} t_1 - u_{min} (T - t_2) && (1) \\
x_T &= x_0 + v_0 t_1 + \frac{1}{2} u_{max} t_1^2 + (v_0 + u_{max} t_1)(t_2 - t_1) \\
&\quad + (v_0 + u_{max} t_1)(T - t_2) - \frac{1}{2} u_{min} (T - t_2)^2 && (2)
\end{aligned}
$$

Furthermore, each boundary arc induces one additional constraint, in that one of the following conditions must hold:

- If the duration of the boundary arc is nonzero $(t_2 - t_1 > 0)$, then x_{t_2} must be on the constraint at the beginning of the boundary arc, $v_{t_1} = v_{max}$.
- If the above condition does not hold, the corresponding arc must vanish $(t_2 - t_1 = 0)$.

$$\begin{aligned} v_T &= v_0 + u_{max}t_1 - u_{min}(T - t_1) && (1) \\ x_T &= x_0 + v_0 t_1 + \frac{1}{2}u_{max}t_1^2 \\ &+ (v_0 + u_{max}t_1)(T - t_1) - \frac{1}{2}u_{min}(T - t_1)^2 && (2) \\ t_2 &= t_1 && (3) \end{aligned}$$

Or

$$\begin{aligned} v_T &= v_0 + u_{max}t_1 - u_{min}(T - t_2) && (1) \\ x_T &= x_0 + v_0 t_1 + \frac{1}{2}u_{max}t_1^2 + (v_0 + u_{max}t_1)(t_2 - t_1) \\ &+ (v_0 + u_{max}t_1)(T - t_2) - \frac{1}{2}u_{min}(T - t_2)^2 && (2) \\ v_{max} &= v_0 + u_{max}t_1 && (3) \end{aligned}$$

4) Solution:

$$\text{when } t_2 = t_1:$$

$$\begin{aligned} t_1 = t_2 &= \frac{-v_0(u_{max} + u_{min}) \pm \sqrt{v_0^2(u_{max} + u_{min})^2 - u_{max}(u_{max} + u_{min})(v_0^2 - v_T^2 + 2u_{min}x_0 - 2u_{min}x_T)}}{u_{max}(u_{max} + u_{min})} \\ &= \frac{-v_0(u_{max} + u_{min}) \pm \sqrt{(u_{max} + u_{min})(v_0^2 u_{min} + v_T^2 u_{max} - 2u_{max}u_{min}x_0 + 2u_{max}u_{min}x_T)}}{u_{max}(u_{max} + u_{min})} \\ T &= \frac{(u_{max} + u_{min})t_1 + (v_0 - v_T)}{u_{min}} \end{aligned}$$

Or

$$
\begin{aligned}
&\text{when } t_2 - t_1 > 0: \\
t_1 &= \frac{v_{max} - v_0}{u_{max}} \\
t_2 &= \frac{u_{min} v_0^2 - 2u_{min} v_0 v_{max} - u_{max} v_{max}^2 + u_{min} v_{max}^2 + u_{max} v_T^2 - 2u_{max} u_{min} x_0 + 2u_{max} u_{min} x_T}{2u_{max} u_{min} v_{max}} \\
T &= \frac{v_0 - v_T + u_{max} t_1}{u_{min}} + t_2
\end{aligned}
$$

$$
u^*(t) = \begin{cases} -u_{min} & 0 \le t < t_1 \\ 0 & t_1 \le t < t_2 \\ +u_{max} & t_2 \le t \le T \end{cases}
$$

$$
\begin{aligned}
v_T &= v_0 - u_{min} t_1 + u_{max}(T - t_1) && (1) \\
x_T &= x_0 + v_0 t_1 - \frac{1}{2} u_{min} t_1^2 \\
&\quad + (v_0 - u_{min} t_1)(T - t_1) + \frac{1}{2} u_{max} (T - t_1)^2 && (2) \\
t_2 &= t_1 && (3)
\end{aligned}
$$

Or

$$
\begin{aligned}
v_T &= v_0 - u_{min} t_1 + u_{max}(T - t_2) && (1) \\
x_T &= x_0 + v_0 t_1 - \frac{1}{2} u_{min} t_1^2 + (v_0 - u_{min} t_1)(t_2 - t_1) \\
&\quad + (v_0 - u_{min} t_1)(T - t_2) + \frac{1}{2} u_{max} (T - t_2)^2 && (2) \\
-v_{min} &= v_0 - u_{min} t_1 && (3)
\end{aligned}
$$

Solution:

$$
\begin{aligned}
&\text{when } t_2 = t_1: \\
t_1 = t_2 &= \frac{v_0(u_{max} + u_{min}) \pm \sqrt{v_0^2(u_{max} + u_{min})^2 - u_{min}(u_{max} + u_{min})(v_0^2 - v_T^2 - 2u_{max} x_0 + 2u_{max} x_T)}}{u_{min}(u_{max} + u_{min})} \\
&= \frac{v_0(u_{max} + u_{min}) \pm \sqrt{(u_{max} + u_{min})(v_0^2 u_{max} + v_T^2 u_{min} + 2u_{min} u_{max} x_0 - 2u_{min} u_{max} x_T)}}{u_{min}(u_{max} + u_{min})} \\
T &= \frac{(u_{min} + u_{max}) t_1 + (v_T - v_0)}{u_{max}}
\end{aligned}
$$

or

$$
\begin{aligned}
t_1 &= \frac{v_0 + v_{min}}{u_{min}} \\
t_2 &= \frac{u_{max} v_0^2 + 2 u_{max} v_0 v_{min} + u_{max} v_{min}^2 - u_{min} v_{min}^2 + u_{min} v_T^2 + 2 u_{max} u_{min} x_0 - 2 u_{max} u_{min} x_T}{2 u_{max} u_{min} v_{min}} \\
T &= \frac{v_T - v_0 + u_{min} t_1}{u_{max}} + t_2
\end{aligned}
$$